Fredholm Alternative for the p-Laplacian and its consequences

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We shall discuss the existence and multiplicity of the solutions to the boundary value problem

$$-\Delta_p u - \lambda|u|^{p-2}u = h(x,u) \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega\,. \tag{1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p > 1$ is a real number, $\lambda \in \mathbb{R}$ is a spectral parameter and $\Delta_p u := \operatorname{div}(|\operatorname{grad} u|^{p-2}\operatorname{grad} u)$. Let $\lambda_1 > 0$ be the principal eigenvalue of $-\Delta_p$ subject to homogeneous Dirichlet boundary conditions with corresponding eigenfunction φ_1. We concentrate on the behaviour of the solutions under the assumption that λ is near λ_1 (and possibly $\lambda = \lambda_1$). Note that for $p = 2$, the problem (1) can be viewed as a sublinear perturbation of some linear problem and so that Lyapunov-Schmidt's method can be used to treat the resonant case $\lambda = \lambda_1$. If $p \neq 2$ the left-hand side of the equation 1 is no longer linear and Lyapunov-Schimdt's method for nonlinear perturbations of linear operators can not be used.

As an alternative, we propose a method based on the bifurcation of the solution pairs $\lambda \in \mathbb{R}$, $u \in W_0^{1,p}(\Omega)$ of (1) when $\lambda \to \lambda_1$ and $\|u\| \to \infty$. It appears that solvability of (1) can be deduced from the asymptotic behaviour of large solutions. On this purpose, we derived an asymptotic formula for large solutions using linearization 1 about λ_1 and φ_1. With this formula at hand, the problem of determining solvability of (1) is reduced to the study of asymptotic behaviour of integrals

$$\int_\Omega h(x,u)\varphi_1\,,$$

where one employs the asymptotic formula for large solutions.

In the new light of results obtained by bifurcation-type arguments, we revise previous results obtained by combination of the method of upper and lower solutions and variational approach.

We will also emphasize importance of the interplay between numerical experiment and development of new theoretical methods on one hand, and theoretical results and development of new numerical methods on the other hand. Finally, we would like to discuss several directions of future research and some open problems as well. Among other ones, this talk has evolved from the following six recent works:

- P. Girg, P. Takáč: Bifurcations of Positive and Negative Continua in Quasilinear Elliptic Eigenvalue Prolems, in Preparation.
- P. Drábek, P. Girg, P. Takáč: Bounded perturbations of homogeneous quasilinear operators using bifurcations from infinity, *J. Diff. Equations* **(204)** (2004) 265–291.
- J. Čepička, P. Drábek, P. Girg: Quasilinear boundary value problems: existence and multiplicity results, to appear in Proceedings of *Variational Methods: Open Problems, Recent Progress, and Numerical Algorithms* held in Flagstaff, 2002.

- P. Drábek, P. Girg, P. Takáč, M. Ulm: The Fredholm Alternative for the p-Laplacian: bifurcation from infinity, existence and multiplicity of solutions, *Indiana Univ. Math. J.* **(53)** 2004 433–482 .
- J.-L. Gámez, P. Girg: Fredholm Alternative for the p-Laplacian at the first eigenvalue and bifurcations from the infinity, Preprint No. 144, University of West Bohemia, 2001.
- P. Drábek, P. Girg, R. Manásevich: Generic Fredholm-type results for the one-dimensional p-Laplacian, *Nonlinear Differ. Equ. and Appl.* **(8)** (2001), pp. 285–298.